A GRH-conditional exclusion for odd perfect numbers with ordinary exponents two or four

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Computational proof for independent review

Abstract

Assuming the generalized Riemann hypothesis for Dirichlet L-functions, any odd perfect number has a prime divisor of positive even exponent at least six. The computer-assisted proof bounds two reciprocal sums over prime inputs whose cyclotomic values are rough. A sieve through 10^{12} and a centered Selberg square on dyadic blocks bound these sums. Exempting one prime from the local tests makes the estimate uniform in the special prime and its exponent. The contradiction has margin greater than 0.00113722 in reciprocal weight.

Let $\sigma(n) = \sum_{d|n} d$. An integer N is perfect if $\sigma(N) = 2N$. Euler's form of an odd perfect number is $N = e^{\alpha} m^2$, where e is prime, $(e, m) = 1$ and $e \equiv \alpha \equiv 1 \pmod 4$. We call e the *Euler* or *special prime*. The other prime divisors are *ordinary*. A *source* p is the base of an ordinary component p^a. A *receiver* is an odd prime divisor r of $\sigma(p^a)$. All logarithms are natural.

Conditional theorem. *Assume the generalized Riemann hypothesis for Dirichlet L-functions. There is no odd perfect number of the form*

$$N = e^{\alpha} \prod_{p \in S} p^2 \prod_{p \in F} p^4,$$

where e is the Euler prime and the displayed primes are pairwise distinct. The assertion holds for every allowed special exponent $\alpha \equiv 1 \pmod 4$. In particular, under GRH some ordinary prime divides any odd perfect number to at least its sixth power.

Earlier work and the least-prime input. Ochem and Rao [5] proved that an odd perfect number exceeds 10^{1500}, and Nielsen [6] proved that it has at least ten distinct prime factors. For restrictions on the ordinary exponents, Steuerwald and Kanold excluded the uniform exponents two and four, respectively (see [1, Sec. 1]). Fletcher, Nielsen and Ochem explicitly identified the mixed case as open in that section. They also sharpened Yamada's effective upper bound on the least prime. We use the following exact form of [1, Main Theorem]. For an odd perfect N with special prime e, its hypothesis is

$$\forall\, p^a \parallel N, \quad p \neq e \implies (3 \mid a+1 \text{ or } 5 \mid a+1).$$

Its conclusion reads "The smallest prime factor of N belongs to the range $10^8 < p < 10^{1000}$." Here p denotes the least prime factor of the whole number. The two allowed ordinary exponents give $a + 1 = 3, 5$ exactly. Hence e and every ordinary divisor exceed 10^8 for every allowed α. In particular $3, 5 \nmid N$.

1 The reduction to two prime sums

Write $P = 10^8$, $U = 10^{12}$, $\lambda = 700$, and $\rho = 997195155/10^9$. For $j = 3, 5$ let

$$\mathcal{A}_j = \{p > P : \ p \text{ prime and every prime } r \mid \Phi_j(p) \text{ satisfies } r \geq p/\lambda\}.$$

Define $W_j(I) = \sum_{p \in I \cap \mathcal{A}_j} (p-1)^{-1}$ and $W(I) = W_5(I) + \rho W_3(I)$. Every prime divisor of a candidate exceeds P by the main theorem of Fletcher, Nielsen and Ochem [1]. For an ordinary component p^a, let $w_p = \log(\sigma(p^a)/p^a)$. Then $1/(p+1) \le w_p < 1/(p-1)$, by the first two terms and the infinite geometric sum.

An ordinary source of exponent $j-1$ contributes the odd divisor sum $\Phi_j(p)$. Every prime r dividing this sum divides N. Since $r > P$, its multiplicative order for the base p is j, so $r \equiv 1 \pmod{j}$. Such an r cannot itself be an ordinary source of exponent $j-1$, for then $j \mid \Phi_j(r)$ would imply $j \mid N$. Thus ordinary receivers of square sources have exponent four, and ordinary receivers of fourth-power sources have exponent two.

Let R_S, R_F denote the sums of w_p over sources in their corresponding $\mathcal{A}_j$, and N_S, N_F the sums over all other sources. Assign each of the latter to one prime receiver $r < p/\lambda$. The source weight is less than $1/(\lambda r)$, since $p - 1 \ge \lambda r$. A receiver r of exponent a takes at most a assignments, because $v_r(\sigma(N)) = v_r(2N) = a$. Hence, with

$$g = \frac{4(P+1)}{\lambda P}, \qquad b = \frac{2(P+1)}{\lambda P},$$

we have $N_S \le g(R_F + N_F) + d_S$ and $N_F \le b(R_S + N_S) + d_F$. Indeed, $a/(\lambda r) \le a(P+1)w_r/(\lambda P)$. For $\alpha = 1$ the Euler prime can take only one assignment, so $d_S + d_F \le 1/P$. Its own log weight is also at most $1/P$.

Put $k = (1+b)/(1+g)$ and $a_0 = (1-gb)/(1+g)$. Multiply the first charging inequality by k and add the second. Using $1 - kg = k - b = a_0$ and the total log abundancy $\log 2$, rearrangement gives

$$R_F + kR_S \ge a_0(\log 2 - 1/P) - 1/P > 0.68919757.$$

Since $k < 0.99715910 < \rho$, this yields $W((P, \infty)) > 0.68919757$.

2 A finite certificate

Every odd input can be written as $2i+1$. For each j and each prime $q \le U/700$, all excluded roots are enumerated as follows. At $q = j$ the root is 1. At $q \equiv 1 \pmod{j}$ they are precisely the $j-1$ nonidentity powers of an element of order j. At other primes there are no roots. Mark $2i+1 \equiv r \pmod{q}$ only for $i \ge 350q$. This implements the strict inequality $700q < 2i+1$. A separate ordinary prime sieve removes composite inputs. The programs sum $\lfloor 10^{18}/(p-1) \rfloor$ for each retained prime. Adding one unit per survivor gives a rational upper enclosure for each sum, including when a prime survives in both roles.

Interval	Prime inputs	Φ_3 survivors	Φ_5 survivors
$(10^8, 10^{10}]$	449,291,056	25,112,811	42,041,606
$(10^{10}, 10^{12}]$	37,152,859,507	1,596,763,246	2,674,724,392

For $H = W((P, U])$ the resulting outward-rounded bounds are

$$0.059345570776 < H < 0.059345575111.$$

A bit-packed segmented sieve reproduced the entire first interval against the original program. On the second interval block endpoints partition the domain, and the prime count agrees with $\pi(10^{12}) - \pi(10^{10})$. Direct polynomial evaluation in the final thousand integers before 10^{11} and 10^{12} independently checks both role counts and integer weights there. The original byte-array method also checks all five integer totals on the full billion-integer blocks ending at 10^{11}, $5 \cdot 10^{11}$ and 10^{12}. The verification archive reproduces these tests. No empirical distribution model enters H.

Arithmetic and reproducibility. The stored columns for each role are its count n_j and its integer sum s_j. They yield

$$\frac{\rho s_3 + s_5}{10^{18}} \;\le\; H \;\le\; \frac{\rho(s_3+n_3)+s_5+n_5}{10^{18}}$$

when aggregated on the whole head. All additions across blocks and the multiplication by ρ are exact rational operations. In generating the roots, the largest modular product is below $(10^{12}/700)^2 < 2^{64}$. Each block mask indexes at most $5 \cdot 10^8$ odd integers in the full run. Within a block, the maximum possible sum of reciprocal units fits in an unsigned 64-bit integer even if every input survives. The prime list includes all primes up to the appropriate root cutoff. The elementary primality sieve requires only primes up to 10^6. The byte-array reference sieve processes odd inputs in smaller blocks. Agreement includes the two integer sums in every overlapping block.

3 The denominator and the principal prime sum

For each of the two orders j, set $m = j-1$. The root counts $\nu(q)$ from the finite sieve give $g_q = \nu(q)/(q-1)$ and $h_q = g_q/(1-g_q)$. Extend h by squarefree multiplicativity, with $h(1)=1$ and zero on other integers. Write $G_j(R) = \sum_{d\le R} h(d)$. On a block $(x, 2x]$, $x = 2^i U$, fix $R = \lfloor x^{1/4}/\sqrt{\log x}\rfloor$. Then $R < x/700$ on every block. The square constructed in Section 4 has mean $1/G_j(R)$.

Lemma 1. *For $R \ge 10^6$, $G_3(R) \ge .6563(\log R + 1.24)$ and $G_5(R) \ge .3882(\log R + 1.42)$. For $R \ge 10^9$ one may instead use*

$$G_j(R) \ge A_j^-(\log R + c_j^- - .245) - E_j R^{-a}, \quad a = 5/16,$$

where (A_j^-, c_j^-, E_j) is $(.6563, 1.647, 68)$ or $(.3882, 2.023, 90)$ respectively.

Proof. Write $\sum h(d)d^{-s} = Z(1+s)H(s)$ initially for $\Re s > 0$, with $Z(z) = \zeta(z)\prod_{\chi\ne\chi_0 \bmod j} L(z,\chi)$. At $p \ne j$ let f be the order of p modulo j and $b = m/f$. At $p = j$ take $f = b = 1$. The Euler factor of H is

$$H_p(s) = (1 + h_p p^{-s})(1 - p^{-f(1+s)})^b.$$

Indeed the character values at p run through the fth roots, each b times. At the ramified prime only the ordinary zeta factor remains. At a split prime the first-order term in p^{-s} has coefficient $h_p - m/p = O(p^{-2})$. At the remaining unramified primes $f \ge 2$. Thus H converges absolutely and uniformly on $\Re s \ge -a$. At zero the Laurent coefficients of the Dirichlet series are $A/s + Ac + O(s)$, where

$$A = H(0)\prod_{\chi\ne\chi_0} L(1,\chi), \quad c = \gamma + \frac{H'(0)}{H(0)} + \sum_{\chi\ne\chi_0}\frac{L'(1,\chi)}{L(1,\chi)}.$$

Real factors at 1 are positive, and conjugate factors pair. The split and ramified logarithmic derivatives vanish since $h_p/(1+h_p) = b/(p-1)$ there; at the other primes the derivative is $bf\log p/(p^f-1) > 0$.

Here are the finite enclosures for the residue and the shifted line. Expand H_p in p^{-s} and sum the absolute coefficients weighted by p^{ak} at degree k, through $p \le 10^5$. At larger split primes put $z = p^{-1+a}$. The absolute linear contribution is at most $21p^{-2+a}$; the higher terms total at most $24z^2$ by binomial expansion and $h_p p^a \le 4.001z$. A nonsplit factor contributes at most $3z^2$ beyond its constant. The sum of n^{-v} beyond N in one class modulo l is at most $N^{-v} + N^{1-v}/(l(v-1))$ for $v > 1$. Apply this with $l = j$ for split terms, $l = 1$ otherwise, and use $1+u \le e^u$. At zero each omitted Euler

factor differs from 1 by at most $45/p^2$; their product exceeds .999. For the logarithmic derivative at zero simply discard its nonnegative tail. Sum the nonprincipal L, L' to 30000, a multiple of j. Partial summation bounds their errors respectively by $j/30000$ and $j\log(30000)/30000$. For approximations l, l' with errors u, v, the ratio error is at most $(v+|l'/l|u)/(|l|-u)$. These procedures give

j	$\sup_{\Re s=-a}\lvert H(s)\rvert$ upper	A lower	c lower
3	24.412	.65636	1.64720
5	13.647	.38823	2.02396

All three displayed columns round in the indicated direction: the first is an upper bound and the last two are lower bounds.

Minorize $1_{d\le R}$ by $(1-(d/R)^{10})^6$ on $d\le R$ and zero elsewhere. Its Mellin kernel is $K(s)=s^{-1}\prod_{k=1}^{6}10k/(s+10k)$ by the beta integral. Shifting the inversion integral to $-a$ crosses the double pole at zero of $R^sK(s)Z(1+s)H(s)$. Since $K(s)=1/s-.245+O(s)$, its residue is $A(\log R+c-.245)$. Euler summation and partial summation as below give polynomial growth on the intervening strip (at height $t\ge 256$, $O((1+t)^a\log(2+t))$ per zeta or L factor suffices). The kernel decays as $|t|^{-7}$, so the horizontal integrals vanish. Conjugate symmetry gives an absolute error at most $R^{-a}\sup|H|I_j$, with

$$I_j=\pi^{-1}\int_0^\infty |K(-a+it)Z(1-a+it)|\,dt.$$

For completeness, the numerical upper integral uses the following enclosures. Set $\sigma=1-a$, $M=1920$, and $s'=\sigma+it$. Approximate each nonprincipal $L(s')$ by its first M terms, with error at most $j|s'|M^{-\sigma}/\sigma$. For $\zeta(s')$ add $M^{1-s'}/(s'-1)$ to its first M terms; the error is at most $|s'|M^{-\sigma}/\sigma$. These follow from bounded character partial sums and Euler summation. On $[0,256]$ use cells of width $1/512$. Evaluate the polynomials and zeta correction at the midpoint. For cell-wide upper moduli add half the cell width times $\sum_{n\le M}n^{-\sigma}\log n$ for each polynomial, and also $M^a(\log M/a+1/a^2)$ times the half-width for the correction. Evaluate the kernel modulus at the left endpoint, where it is greatest, and use the right endpoint in the truncation bounds.

One implementation rotates integer complex pairs at scale $Q=2^{46}$. The initial phases and step rotations use 18 Taylor terms with argument below .015; quantization costs at most $2/Q$ in Euclidean norm. Products use signed 128-bit intermediates. After n steps the error is at most $(2+4n)(1+2/Q)^n/Q<8\cdot10^{-9}$ for $n\le 131072$, by the triangle inequality. The sum over 1920 vectors and the separate correction vector therefore admit an absolute .00005 guard per midpoint modulus. The derivative bounds get an extra .001. At $t\ge256$ truncate instead at $j\lceil t\rceil$; each factor is at most C_jt^a, where

$$C_j=[j(1+1/256)]^a(1/a+1/256)+(1+\sigma/256)j^a/\sigma.$$

This follows from $\sum_{n\le M}n^{-\sigma}\le M^a/a$ and the preceding remainder bounds. The remaining integral is bounded by $6!10^6C_j^m256^{am-6}/(6-am)$ before division by π. The resulting I_j upper bounds, including both ranges, are 2.670416 and 6.294169. The fixed-point implementation uses 48-digit decimal quadrature and adds 10^{-15} after accumulation. The independently implemented midpoint quadrature also meets the caps 2.70, 6.40. Using $\sup|H|<25,14$ yields $E_j=68,90$.

At 10^9 the curved bounds exceed the asserted affine lines by .00160, .00038, respectively, with increasing surplus thereafter. For the interval from 10^6 to 10^9 enumerate squarefree active products in increasing order of their prime factors, starting from 10^{16} units and rounding down after each multiplication by h_q. There is exactly one branch per product. Compare cumulative lower credit at each left endpoint to the affine line at the next endpoint on a grid advancing from n by $\lfloor n/64\rfloor$, with 10^9 included. The minimum slacks exceed .271, .243. Additional grid points supply lower credits

at all smaller frozen radii needed for integration. The bound $h_q \le 1$ and at most 30 factors per integer in this range bound cumulative rounding loss by $32R/10^{16}$. This also gives upper credits where needed below. □

Principal integration. Put $T = \log U$, $h = \log 2$, $t_i = T + ih$, and let $a(y) = \rho/G_3(R_i) + 1/G_5(R_i)$ on block i. Its continuous coefficient is $T\int_U^\infty a(y)\,dy/((y-1)\log y)$. On block i use the two lower denominator credits and the factor

$$T\log(1+h/t_i)/(1-e^{-t_i}).$$

Through $i = 92$ use the enumerated credits. Thereafter use the curved lower bound, lowering $\log R$ by evaluating $\log(x^{1/4}/\sqrt{\log x} - 1)$ with a downward guard. From the first endpoint $t_n \ge 8192$ onward, every point of log height v has $\log R \ge (v-h)/4 - (\log v)/2 - 2\cdot 10^{-9} \ge .248v$; the ratio of the middle expression to v increases. Integrating v^{-2} gives the remainder

$$\frac{T(\rho/.6563 + 1/.3882)}{.248\,t_n(1-e^{-t_n})}.$$

The three successive coefficient contributions are below 11.511294, 4.971301, .055699. Before outward rounding by ranges their total is below 16.538293; the independent computation stopping at log height 4096 gives 16.538627. Use the common ceiling 16.539.

To replace dy by $d\theta(y)$ use $v(y) = a(y)/((y-1)\log y)$, positive and nonincreasing. By Broadbent et al. [3], $\theta(y) \le (1+\delta)y$, $\delta = 2\cdot 10^{-8}$. Summing the two reduced classes modulo 3 in [4, Thm. 1.2] gives $\theta(U) \ge U(1 - 2/(840T))$. Thus

$$\theta(y) - \theta(U) \le (1+\delta)(y-U) + U(\delta + 2/(840T)).$$

Integration against $-dv$, including its downward jumps (the dyadic endpoints are composite), increases the continuous coefficient by at most $16.539\delta + 2U(\delta + 2/(840T))/(U-1) < .000174$, using $a(U+) \le 2$. The unconditional principal coefficient is less than 16.540.

4 The centered square and its coefficient mass

On the reduced residues at an active q, let I_q indicate the roots and put $f_q = (g_q - I_q)/(1-g_q)$. Its values off and on the roots are $h_q, -1$. Its mean is zero and its mean square is h_q. For squarefree active d put $f_d = \prod_{q\mid d} f_q$, including $f_1 = 1$, and $D_R = \sum_{d\le R} f_d$. On a survivor $D_R = G_j(R)$ because $R < x/700$. Hence $(D_R/G_j(R))^2$ majorizes the indicator. The Chinese remainder theorem gives orthogonality of distinct f_d and mean $h(d)$ for f_d^2. The principal coefficient of the square is therefore $1/G_j(R)$. Each character in it has conductor dividing a least common multiple of two indices, hence at most R^2. On the block all input primes exceed the sieve primes, so induced characters may be replaced by their primitive representatives.

Use normalized Fourier coefficients and their sum of absolute values as norm. For a nonprincipal local character, the coefficient of f_q has magnitude h_q times the absolute mean of the character on the roots. At a split prime the roots are the nonidentity elements of the order-j subgroup. That mean is 1 for the $(q-1)/j$ characters trivial on the subgroup and $-1/(j-1)$ for all others. Omit the principal character when summing the magnitudes. At the ramified prime the single root is 1. Thus the local norm is

$$w(q) = \frac{(j-1)(2(q-1)-j)}{j(q-j)} \quad (q \equiv 1 \bmod j), \qquad w(j) = 1.$$

Extend w by squarefree multiplicativity, zero off support, and put $V(u) = \sum_{n \le u} w(n)$. Products on independent factors multiply their norms. Convolution and the triangle inequality give $\|D_R^2/G_R^2\|_1 \le V(R)^2/G_R^2$, where $G_R = G_j(R)$. Projection onto nonprincipal coefficients only reduces the norm. Embed coefficient vectors on the common product of local groups before comparing adjacent radii. For $L < R$ write $D_R^2 - D_L^2 = (D_R - D_L)(D_R + D_L)$. The norm of the change between the normalized squares is at most

$$J(L,R) = \frac{V(R)^2 - V(L)^2}{G_R^2} + V(L)^2 \left(\frac{1}{G_L^2} - \frac{1}{G_R^2} \right). \tag{1}$$

Indeed the difference of the unnormalized kernels has norm at most $V(R) - V(L)$ before squaring, and changing the old denominator gives the second term. This estimate permits summation by parts across a shared endpoint with a single character charge.

Lemma 2. *For $R \ge 877000000$ the sum of the two norm bounds, weighted by $\rho, 1$, is at most $CR^2/(\log R)^2$, where $C = \rho(.175/.6563)^2 + (.067/.3882)^2 < .100689$.*

Proof. Write $Q = 1000$, $Y = 877000$, $X = QY$. Take $(c,d) = (.175, .70)$ or $(.067, .435)$ for $j = 3, 5$. We first bound $\sum_{p \le y} w(p) \log p \le dy$ for $y \ge Q$. Enumeration of prime jumps through 10^8 gives respective upper ratios $.666637, .402743$. The same calculation gives $S = \sum_{p \le Q} w(p) \log(p)/p \le 3.506, 2.199$. The logarithms are bounded above on 256 equal relative subintervals of each dyadic prime range, rounding their right endpoint logarithms up in units of 10^{-7}. For larger y put $b = 2(j-1)/j$. At split primes $0 \le w(p) - b \le 5/p$. The sum in question is at most

$$b\theta(y; j, 1) + \log j + 5(1 + \log^2 y).$$

Here $\sum_{n \le y} \log(n)/n \le 1 + \log^2 y$ by integral comparison. For $j = 3$, [1, Sec. 8.4] gives $|\theta(y; 3, 1) - y/2| \le .262y/\log y$. For $j = 5$, use [1, Prop. 5] up to 10^{10}, where the error is at most $1.412480\sqrt{y}$, and [4, Thm. 1.2] from $8 \cdot 10^9$ onward, where it is at most $y/(840 \log y)$. Thus $.010y/\log y$ suffices for $j = 5$ throughout $y \ge 10^8$. Substituting the lower endpoint gives total relative bounds $.685649, .400887$ and the errors decrease thereafter.

We prove $V(u) \le cu$ for $u \ge Y$. Enumerate all active squarefree products through X as for the denominator, enclosing each multiplication by $w(q)$ between floor and ceiling in units of 10^{-8}. Compare the upper V at the right of each grid interval against c times the left, advancing by $\lfloor u/256 \rfloor$ and including X. This verifies the base $[Y, X]$. With $H_w(u) = \sum_{m \le u} w(m)/m$, termwise upward rounding gives $H_w(Y) \le 3.826, 2.296$. For integer $x > X$, use

$$V(x) \log x = \int_1^x V(u) \frac{du}{u} + \sum_{n \le x} w(n) \log n.$$

The integral is at most $cx + Y H_w(Y)$ by induction and $\log(Y/n) \le Y/n$. In the second sum extract a prime divisor and drop coprimality. Primes through Q contribute at most cxS since $Y \le x/p < x$. For $p > Q$ sum first over the complementary factor $m \le x/Q$, obtaining at most $dx H_w(x/Q)$. By partial summation,

$$H_w(x/Q) \le H_w(Y) + c - V(Y)/Y + c \log(x/X).$$

It suffices that $c(\log X - 1 - S) - H_w(Y)/Q - d(H_w(Y) + c - V(Y)/Y) > 0$, because the surplus at x increases by $c(1-d)\log(x/X)$. Using the integer lower credit for $V(Y)$ and the upper credits above gives initial surpluses $> .131$ and $> .163$. This proves the induction for integers and hence real arguments. The denominator lemma gives the claimed C.

In the finite enumerations $w(q) \le 2$ and every index below 10^9 has fewer than ten prime factors. Per-index upward loss at scale 10^8 is below 1024 units. Since $\sum_{d \le R} 2^{\omega(d)} \le \sum_{d \le R} \tau(d) \le R(1 + \log R)$, the accumulator totals fit unsigned 64-bit words; local products use 128 bits. □

5 Character bounds and integration

Write $\psi(y,\chi)=\sum_{n\le y}\chi(n)\Lambda(n)$ and $\theta(y,\chi)=\sum_{p\le y}\chi(p)\log p$. Under Dirichlet GRH the journal version of Lee [2, Thm. 4.2] bounds $|\psi(y,\chi)|$ for χ nonprincipal, $y\ge x_0\ge\max(e^{10},q)$. For $t=\log y$, its bound divided by $\sqrt{y}\,t^2$ is

$$\frac{1}{8\pi}+\frac{\log q/(2\pi)+\Omega_0}{t}+\frac{\Omega_1}{t^2}+\frac{\Omega_2 e^{-t/2}}{t^2}.$$

For $y\ge U$, [3] bounds $|\psi-\theta|/(\sqrt{y}\,t)$ by $(1+2\cdot10^{-8})(1/t+e^{-t/6}/\log 2)<.051$. Indeed squares contribute at most $\theta(\sqrt{y})$ and each remaining exponent contributes at most $\theta(y^{1/3})$. Here are the journal's Table 5 inputs and the upward coefficient b_0 actually used for $\Omega_0+.051$.

$\log x_0$	Ω_0	Ω_1	Ω_2	b_0
20	5.81532	.0207932	$-3.15402\cdot10^{6}$	6.33542
30	5.17628	.000312938	$-8.10901\cdot10^{8}$	5.66832
40	4.83973	.00000373481	$-1.64282\cdot10^{11}$	5.31911
50	4.64381	$3.92334\cdot10^{-8}$	$-3.10314\cdot10^{13}$	5.11581
60	4.50191	$3.80107\cdot10^{-10}$	$-5.59133\cdot10^{15}$	4.96901

The journal corrects the short-interval smoothing step in arXiv:2301.13457v2. Use the journal equation (23), including the unit in $1+k_3$.

Uniformity for unbounded conductors. Set $t_*=T+93h>92$. For primitive nonprincipal χ, $q\le R^2\le\sqrt{y}$, and $t\ge t_*$, we verify $|\theta(y,\chi)|\le.18\sqrt{y}\,t^2$. The uniform far-tail estimate uses an intermediate step in Lee's journal proof [2, eq. (28)], applying Theorem 2.4 with $h_y=\sqrt{y}\log y$ without the conductor restriction $q\le x_0$ imposed in equation (31). Equation (28) requires $y\ge x_0\ge e^{10}$ and $q\ge3$. Thus Table 3 at $x_0=e^{80}$ gives $k_1=.37637$, $k_2=2.22565$ for every conductor here. The smoothed ψ is bounded by

$$[t/(8\pi)+\log q/(2\pi)+k_1+y^{-1/2}+y^{-1}]\sqrt{y}\,t+k_2\sqrt{y}\log q+g_2(q).$$

Both branches of the journal's $g_2(q)$ are bounded by $317.501+.593(\log\log q)(\log q)^2+.0758\sqrt{q}\log q+2.751\log q$. This is precisely where (31) uses $q\le y$; here $\log q\le t/2$. The exponentially small terms combined, in units of $\sqrt{y}\,t$, are at most

$$(3.751+.593t\log t+317.501/t)e^{-t/2}+e^{-t}<.001.$$

For smoothing, Table 4 at $x_0=e^{20}$ gives $k_3=1.11929$ in Theorem 3.2. Apply (23), dropping the subtractive k_4. Converting to θ as above now bounds the normalized error by

$$\frac{3}{8\pi}+\frac{.37637+2.22565/2+.0758+.001+1+1.11929+.051}{t_*}<.161<.18.$$

Thus the computation uses only the more conservative .18. The derivation explains the intermediate step underlying (31) and does not extrapolate Table 5 to arbitrary moduli.

The first 93 blocks. Write $v(y)=1/((y-1)\log y)$ and $m_i=\sum_j z_jV_j(R_i)^2/G_j(R_i)^2$ with $(z_3,z_5)=(\rho,1)$. Put $J_0=m_0$ and $J_i=\sum_j z_jJ_j(R_{i-1},R_i)$ for $i>0$. In summing $v\,d\theta(\cdot,\chi)$, partial summation gives boundary differences and an integral against $-dv$. At a shared boundary the old

right endpoint and new left endpoint involve the difference of the coefficient vectors, giving (1). The maximum conductor of either vector is R_i^2. The derivative in log height is $-d\log v(e^t)/dt = 1/(1-e^{-t}) + 1/t$.

For $l \le t \le u$ on block i define

$$B_i(l,u) = \frac{le^{-l/2}}{1-e^{-l}} \min_{2\log R_i \le \log x_0 \le l} \left[\frac{1}{8\pi} + \frac{2\log R_i/(2\pi) + b_0}{l} + \frac{\Omega_1}{l^2} + \frac{\Omega_2 e^{-u/2}}{u^2} \right].$$

Minimize over the five listed rows. This bounds $v(y)|\theta(y,\chi)|$ because $te^{-t/2}/(1-e^{-t})$ decreases. The negative last term must be evaluated at u. Divide each block into 16 equal intervals $[l,u]$ in log height. In coefficient units its contribution is at most

$$TJ_iB_i(t_i,t_i) + Tm_i \sum_{[l,u]} (u-l)B_i(l,u) \left[(1-e^{-l})^{-1} + 1/l \right].$$

Compute G with lower and upper credits as in Lemma 1, and V with the two-sided recursion in Lemma 2. In the first numerator of (1) use upper $V(R)$ and lower $V(L)$; in the second use upper $V(L)$, lower G_L and upper G_R inside the subtraction. For $i = 0, \dots, 92$ the total is $< .469491$. Charge the remaining right endpoint from the last block once using $Tm_{92}B_{92}(t_*,t_*) < .000459$.

The rest of the integral. For each left height $t = t_* + ih$, $\log R \ge t/4 - (\log t)/2 - 2/(e^{t/4}/\sqrt{t}) \ge .225t$. The lower ratio increases and $R \ge 877000000$ there. Lemma 2 bounds the mass by $Ce^{t/2}/[t(.225t)^2]$. At offset $s \in [0,h]$, $v(e^{t+s})\sqrt{e^{t+s}}(t+s)^2 \le (1+10^{-25})te^{-t/2}(1+h/t_*)e^{-s/2}$; omit $1+h/t_*$ at $s=0$. Charge both endpoints separately on every such block. With $r = e^{-h/2}$ the resulting factor for the two endpoints and the integral is

$$F = (1+10^{-25})\left[1 + r(1+h/t_*) + 2(1-r)(1+h/t_*)(1+10^{-17}+1/t_*)\right] < 2.310.$$

The integral uses $\int_0^h e^{-s/2}ds = 2(1-r)$. By integral comparison $\sum_{i\ge 0}(t_*+ih)^{-2} \le t_*^{-2} + 1/(ht_*)$. The total coefficient is bounded by

$$\frac{TFC(.18)}{.225^2}\left[t_*^{-2} + \frac{1}{ht_*}\right] < .361.$$

Both terminal terms at the change of method have now been paid. Adding the bounds gives

$$W((U,\infty)) < [16.540 + .470 + .000459 + .361]/T < 17.372/T.$$

6 Deleting the Euler prime

For each fixed prime $e > P$, let $\mathcal{A}_{j,e}$ be defined as $\mathcal{A}_j$ with the condition at e omitted. An input qualifies if every prime divisor of $\Phi_j(p)$ other than e is at least $p/700$. Define W_e with these enlarged sets and the same weights. Any ordinary source outside its enlarged set has a small *ordinary* receiver. Its assignment has the bounds of Section 1 with $d_S = d_F = 0$, for every special exponent. The contribution of e^α itself is less than $1/(e-1) \le 1/P$. Thus the same weighted addition requires $W_e((P,\infty)) > 0.68919757$.

Finite cost. Each new head survivor must have $e \mid \Phi_j(p)$ and $p > 700e$. There are at most two root classes for $j = 3$ and four for $j = 5$. Each class contains at most $\lfloor U/e \rfloor + 1$ inputs, each of weight at most $1/(700e)$. Consequently $W_e((P,U]) - H \le 6(\lfloor U/P \rfloor + 1)/(700P) < 10^{-6}$, uniformly in e.

Tail cost. For one role let G', V' denote the positive sums of Sections 3 and 4 after dropping divisors divisible by e. With $D_0 = 1 + 5 \cdot 10^{-8}$ we have

$$G'(R) \le G(R) \le (1 + h_e)G'(R) \le D_0 G'(R),$$

since the omitted sum is $h_e G'(R/e)$, interpreted as zero below 1, and $h_e \le 4/(e-5)$. At an inactive prime take $h_e = 0$. The centered-square argument applies on the smaller product of local unit groups. At each radius $V' \le V$, so the coefficient norm bound increases by at most D_0^2. The principal coefficient, weighted against a positive prime measure, increases by at most D_0. The first term of the shared-endpoint inequality increases by at most D_0^2, since both $V'(R) - V'(L)$ and $V'(R) + V'(L)$ are bounded by their undeleted counterparts. For the second term write

$$\frac{1}{G'(L)^2} - \frac{1}{G'(R)^2} = \frac{(G'(R) - G'(L))(G'(R) + G'(L))}{G'(R)^2 G'(L)^2} \le D_0^4 \left(\frac{1}{G(L)^2} - \frac{1}{G(R)^2} \right).$$

Both numerator factors are bounded by their undeleted counterparts by positivity. Removing a local factor preserves the conductor bound R^2. Thus each positive majorant entering the full tail certificate increases by at most D_0^4, including the terminal endpoint and the infinite remainder. This compares the explicit upper bounds; no cancellation of deleted Fourier coefficients is assumed.

Combining these estimates yields, for every $e > P$,

$$W_e((P, \infty)) < 0.059345575111 + 10^{-6} + \frac{D_0^4 \cdot 17.372}{\log U} < 0.68806035 < 0.68919757.$$

This contradicts the lower bound for every special exponent and proves the theorem. The gap in reciprocal weight exceeds 0.00113722. Multiplying by $\log U$ gives about 0.0314 coefficient units, or 0.18% of the tail coefficient 17.372.

Computational materials and AI disclosure

The downloadable review archive (ZIP) contains the manuscript and source, both contour implementations, the head sieves, per-block integer sums and a SHA-256 manifest. Its README gives commands for checking the individual numerical lemmas and running a separate head implementation. No modeled roughness density enters the proof.

Eric B. Schorvitz directed this investigation. OpenAI ChatGPT and Codex assisted in developing the arguments, writing and running the verification programs, checking sources and drafting the manuscript. The author independently rebuilt and reran the head through 10^{10} and reviewed the charging and Euler-prime arguments. The full 10^{12} enumeration was performed in the AI-assisted project environment. External full-range reproduction and peer review remain outstanding.

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